THE SYMPLECTIC IDEAL AND A DOUBLE CENTRALISER THEOREM

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SUMMARY. We interpret a result of S. Oehms as a statement about the symplectic ideal. We use this result to prove a double centraliser theorem for the symplectic group acting on $\bigoplus_{r=0}^{s}\otimes^r V$, where V is the natural module for the symplectic group. This result was obtained in characteristic zero by H. Weyl. Furthermore we use this to extend to arbitrary connected reductive groups G with simply connected derived group the earlier result of the author that the algebra $K[G]^{\mathfrak{g}}$ of infinitesimal invariants in the algebra of regular functions on G is a unique factorisation domain.

INTRODUCTION

Throughout this note k denotes an infinite field, K denotes the algebraic closure of k, n and s are positive integers, V is the vector space k^n and $\mathrm{Mat}_n = \mathrm{Mat}_n(k)$ is the k-algebra of $n \times n$-matrices acting on V as vector space endomorphisms. We denote the vector space $\bigoplus_{r=0}^{s}\otimes^r V$ by $T^{\le s}(V)$. A matrix $u \in \mathrm{Mat}_n$ acts on $\otimes^s V$ by $u(x_1 \otimes \cdots \otimes x_s) = u(x_1) \otimes \cdots \otimes u(x_s)$. For a subset X of Mat_n we denote by $\mathcal{E}^s(X)$ and $\mathcal{E}^{\le s}(X)$ the *enveloping algebra* of X in $\mathrm{End}_k(\otimes^s V)$ and $\mathrm{End}_k(T^{\le s}(V))$ respectively, that is, the subalgebra generated by the endomorphisms corresponding to the elements of X. Let A be an associative k-algebra. The *centraliser algebra*[1] of a subalgebra B of A is defined as the subalgebra of A that consists of the elements of A that commute with all elements of B. We say that the *double centraliser theorem* holds for a subalgebra B of A if B is equal to the centraliser algebra of its centraliser algebra. We say that the double centraliser theorem holds for a subset X of Mat_n acting on $\otimes^s V$ or $T^{\le s}(V)$ if it holds for the corresponding enveloping algebra.

Now assume that $k = K = \mathbb{C}$ and let G be a subgroup of $\mathrm{GL}_n(\mathbb{C})$. In [3] Brauer showed that, if the double centraliser theorem holds for G, one can determine defining equations for $\mathcal{E}^s(G)$ by first determining the centraliser algebra. Determining the centraliser algebra turned out to be equivalent to a version of the first fundamental theorem of invariant theory for G.[2] His methods applied to any semisimple complex Lie subgroup G of $\mathrm{GL}_n(\mathbb{C})$ for which there is a first fundamental theorem of invariant theory, in particular to the symplectic group $\mathrm{Sp}_{2m}(\mathbb{C})$. In these cases we have namely that all finite dimensional Lie group

[1] also called *commutating algebra, commutator algebra* or *commuting algebra.*

[2] In the case $G = \mathrm{GL}_n$ the centraliser algebra was already known and therefore yielded another proof of this version of the first fundamental theorem for GL_n. In characteristic zero one can deduce from this the general first fundamental theorem for GL_n. See [18] p.139.

representations of G are semisimple and the double centraliser theorem holds for semisimple subalgebras of $\mathrm{Mat}_n(\mathbb{C})$.

Let I be the ideal of polynomial functions on Mat_{2m} that vanish on Sp_{2m}. In [18] Weyl showed that finding generators $g_1, \dots, g_r$ of I which have the property that any $f \in I$ can be expressed as $f = \sum_i a_i g_i$ with $\deg(a_i g_i) \leq \deg(f)$ for certain polynomial functions $a_1, \dots, a_r$ on Mat_{2m}, is equivalent to determining defining equations for $\mathcal{E}^{\leq s}(\mathrm{Sp}_{2m})$, for s arbitrary. Then he used Brauers method, which also applies to $T^{\leq s}(V)$, to determine such generators of I.

For the classical groups the first fundamental theorem of invariant theory has been generalised to positive characteristic by De Concini and Procesi in [6], but the algebras $\mathcal{E}^s(G)$ and $\mathcal{E}^{\leq s}(G)$, G classical, are no longer semisimple for all s in positive characteristic. In Section 1 and 3 of this note we will reverse Weyls procedure and show that we can easily derive the double centraliser theorem for Sp_{2m} acting on $T^{\leq s}(V)$ once we have generators for the symplectic ideal with the abovementioned property. All this relies on work of Oehms, De Concini and Procesi and Berele.

In Section 2 we derive another consequence from the results on the symplectic ideal in Section 1. We show that for a connected reductive group G with simply connected derived group, the algebra $K[G]^{\mathfrak{g}}$ of infinitesimal invariants in the algebra of regular functions on G is a unique factorisation domain.

1. The symplectic ideal and the symplectic enveloping algebra

We begin by considering the following property of a set of generators $g_1, \dots, g_r$ of an ideal I in the polynomial ring $k[x_1, \dots, x_n]$:

Any $f \in I$ can be expressed as

$$f = \sum_{i=1}^{r} a_i g_i \textit{ with } \deg(a_i g_i) \leq \deg(f) \tag{1}$$

for certain $a_1, \dots, a_r \in k[x_1, \dots, x_n]$.

This property is related to the associated graded ideal for which we now introduce some notation. Let A be a commutative algebra over k with a filtration $A_0 \subseteq A_1 \subseteq A_2 \cdots$. The associated graded algebra is denoted by $\mathrm{gr}(A)$. If $x \in A_i \setminus A_{i-1}$, then we put $\deg(x) = i$ and $\mathrm{gr}(x) = x + A_{i-1} \in \mathrm{gr}(A)^i = A_i/A_{i-1}$. For an ideal I of A we have $\mathrm{gr}(A/I) \cong \mathrm{gr}(A)/\mathrm{gr}(I)$; see [2], Chapter 3, § 2.4. In particular, when A is graded, we have $\mathrm{gr}(A/I) \cong A/\mathrm{gr}(I)$.

Lemma 1. *Let I be an ideal of the polynomial ring $k[x_1, \dots, x_n]$ which is generated by the nonzero elements $g_1, \dots, g_r$. Then these generators have property* (1) *if and only if the elements* $\mathrm{gr}(g_1), \dots, \mathrm{gr}(g_r)$ *generate the ideal* $\mathrm{gr}(I)$.

Proof. Assume that $g_1, \dots, g_r$ is set of generators of I which have property (1) and let $\mathrm{gr}(f)$ be a nonzero homogeneous element of $\mathrm{gr}(I)$. Then $f = \sum_{i=1}^r a_i g_i$ with $\deg(a_i g_i) \leq \deg(f)$ for certain $a_1, \dots, a_r \in k[x_1, \dots, x_n]$. So $\mathrm{gr}(f) = \sum_i \mathrm{gr}(a_i)\mathrm{gr}(g_i)$ where the sum is over all indices i such that $\deg(a_i g_i) = \deg(f)$. Thus the $\mathrm{gr}(g_i)$ generate $\mathrm{gr}(I)$. Now assume that the latter is the case and let f be a nonzero element of I. We can write $\mathrm{gr}(f) = \sum_{i=1}^r a_i \mathrm{gr}(g_i)$ where a_i is

homogeneous of degree $\deg(f)-\deg(g_i)$. Then $f-\sum_{i=1}^r a_i g_i$ is of strictly lower degree than f and we can finish by induction. $\square$

The symmetric group Sym_r on s symbols acts on $\otimes^r V$ via $\pi\cdot(x_1\otimes\cdots\otimes x_r) = x_{(\pi^{-1})_1}\otimes\cdots\otimes x_{(\pi^{-1})_r}$. Denote the group algebra of Sym_r over k by $k\langle\mathrm{Sym}_r\rangle$. The elements of $\mathrm{End}_{k\langle\mathrm{Sym}_r\rangle}(\otimes^r V)$, i.e. the k-linear endomorphisms of $\otimes^r V$ commuting with the action of Sym_r, were called *bisymmetric substitutions* by Weyl. Clearly all enveloping algebras $\mathcal{E}^r(X)$ consist of bisymmetric substitutions.

To state the proposition below correctly we need some auxiliary notation. For a subset S of a fixed finite dimensional k-vector space W we denote the k-algebra of *polynomial functions* on S, i.e. functions on S that are restrictions of polynomial functions on W, by $k[S]$. The algebra $k[W]$ is a polynomial algebra in the elements of any basis of W^*, because k is infinite. In particular, $k[\mathrm{Mat}_n]$ is a polynomial algebra in the matrix entry functions.

Now take $W=\mathrm{Mat}_n$. Clearly $k[S]$ is isomorphic to $k[\mathrm{Mat}_n]/I$, where I is the ideal of polynomial functions on Mat_n that vanish on S. The algebra $k[S]$ inherits a filtration from $k[\mathrm{Mat}_n]$. We denote the filtration subspace of index s by $k[S]^{\le s}$. If S is closed under multiplication by nonzero scalars, then the ideal I will be homogeneous and $k[S]$ is a graded algebra. We denote the graded subspace of index s by $k[S]^s$. If S is a submonoid of Mat_n, then I is a biideal and $k[S]$ is a k-bialgebra. The subspace $k[S]^{\le s}$ is then a subcoalgebra of $k[S]$ and if S contains the nonzero multiples of the identity, then $k[S]^s$ is a subcoalgebra of $k[S]$. We denote the closure of a subset S of Mat_n under multiplication by nonzero scalars by $k^\times S$. Note that in this notation $k[\mathrm{GL}_n]$ is equal to $k[\mathrm{Mat}_n]$ and not to $k[\mathrm{Mat}_n][\det^{-1}]$.

After Proposition 1 we will only use this notation in the following situation where it is in accordance with the standard notation. If X is a k-defined closed subvariety of $W(K):=K\otimes_k W$ such that the set S of k-defined points in X is dense in X, then $k[S]$ in the notation here is naturally isomorphic to the algebra $k[X]$ of regular functions on X that are defined over k. This applies for example to the symplectic group Sp_n as we shall see later. The next proposition is essentially due to Schur and Weyl.

Proposition 1. *Let M be a submonoid of Mat_n. Then*

(i) the natural map $\mathcal{E}^s(M)^*\to k[k^\times M]^s$ is an isomorphism of coalgebras.
(ii) the natural map $\mathcal{E}^{\le s}(M)^*\to k[M]^{\le s}$ is an isomorphism of coalgebras.
(iii) $\mathcal{E}^s(\mathrm{GL}_n)=\mathrm{End}_{k\langle\mathrm{Sym}_s\rangle}(\otimes^s V)$.
(iv) $\mathcal{E}^{\le s}(\mathrm{GL}_n)=\mathrm{End}_{k\langle\mathfrak{S}\rangle}(T^{\le s}(V))$, where $\mathfrak{S}=1\times 1\times\mathrm{Sym}_2\times\cdots\times\mathrm{Sym}_s$.

Proof. The natural map in (ii) composes $f\in\mathcal{E}^{\le s}(M)^*$ with the monoid homomorphism $M\to\mathcal{E}^{\le s}(M)$. The result is clearly a polynomial function on M of degree $\le s$. That the natural map is injective follows from the fact that endomorphisms of $T^{\le s}(V)$ that represent the elements of M span $\mathcal{E}^{\le s}(M)$, since M is a submonoid of Mat_n.

The algebra $\mathcal{E}^s(M)$ is equal to $\mathcal{E}^s(k^\times M)$, since $\alpha{\cdot}\mathrm{id}\in\mathrm{Mat}_n$ is represented by $\alpha^s{\cdot}\mathrm{id}$ on $\otimes^s V$. So in the case of (i) we may assume that M contains the nonzero multiples of the identity. The injectivity of the natural map, which composes $f\in\mathcal{E}^s(M)^*$ with $M\to\mathcal{E}^s(M)$, now follows as in (ii).

By considering the commutative diagram below and its graded version in case M contains the multiples of the identity, we see that it is now sufficient to prove (i) and (ii) for $M = \mathrm{Mat}_n$, since the two vertical arrows are surjective morphisms of coalgebras.

$$\begin{tikzcd} \mathcal{E}^{\leq s}(\mathrm{Mat}_n)^* \arrow[r] \arrow[d] & k[\mathrm{Mat}_n]^{\leq s} \arrow[d] \\ \mathcal{E}^{\leq s}(M)^* \arrow[r] & k[M]^{\leq s} \end{tikzcd} \tag{2}$$

We first consider the natural map of (i) for $M = \mathrm{GL}_n$. To avoid confusion, we will write $k[\mathrm{Mat}_n]$ instead of $k[\mathrm{GL}_n]$, since these algebras are canonically isomorphic. It comes from the natural map $\mathrm{End}_k(\otimes^s V)^* \to k[\mathrm{Mat}_n]^s$. Let $I = I(n,s) = \{1,\dots,n\}^s$ as in [11, 2.1]. We have a standard basis $(E_{i,j})_{(i,j)\in I\times I}$ of $\mathrm{End}_k(\otimes^s V)$. Our map, maps the element $E^*_{i,j}$ of the dual basis to Green's $c_{i,j}$. The $c_{i,j}$ form a basis when we pass to $(I\times I)/\!\sim$, where $\sim$ is the equivalence relation given by the action of the Sym_s on $I\times I$ defined in [11, 2.1]. Now we form the dual basis $c^*_{\overline{(i,j)}}$ and define $c^*_{i,j} = c^*_{\overline{(i,j)}}$. Here $\overline{(i,j)}$ denotes the canonical image of (i,j) in $(I\times I)/\!\sim$. Then the transpose of our map maps $c^*_{i,j}$ to $\sum_{(k,l)\sim(i,j)} E_{k,l}$, which is precisely what Green's map $(k[\mathrm{Mat}_n]^s)^* \to \mathrm{End}_k(\otimes^s V)$ does. Assertion (i) for GL_n and (iii) now follow from [11, 2.6]. Note that (iii) shows that $\mathcal{E}^s(\mathrm{Mat}_n) = \mathcal{E}^s(\mathrm{GL}_n)$.

Now we consider the natural map of (ii) for $M = \mathrm{GL}_n$. We have seen that it is injective. It is also surjective, since for any polynomial function f on GL_n of degree $\leq s$ we can find an element of $\mathrm{End}(T^{\leq s}(V))^* \cong T^{\leq s}(\mathrm{Mat}_n^*)$ that is mapped to f. We have $\mathcal{E}^{\leq s}(\mathrm{GL}_n) \subseteq \bigoplus_{r=0}^s \mathcal{E}^r(\mathrm{GL}_n)$. Since the natural maps of (i) and (ii) for $M = \mathrm{GL}_n$ are bijective we must have equality. It is now also clear that (ii) is an isomorphism of coalgebras.

Since the image of $k\langle\mathfrak{S}\rangle$ in $\mathrm{End}_k(T^{\leq s}(V))$ contains the projections given by the direct sum decomposition of $T^{\leq s}(V)$, we have that $\mathrm{End}_{k\langle\mathfrak{S}\rangle}(T^{\leq s}(V))$ is equal to $\bigoplus_{r=0}^s \mathrm{End}_{k\langle\mathrm{Sym}_r\rangle}(\otimes^r V)$: it consists of $(s+1)$-tuples of "bisymmetric substitutions". So (iv) now follows from (iii) and it is now also clear that $\mathcal{E}^{\leq s}(\mathrm{GL}_n) = \mathcal{E}^{\leq s}(\mathrm{Mat}_n)$. □

Remarks 1. 1. Assertion (iv) of Proposition 1 was proved in a different way by Weyl, see [18, Thm. 4.4.E].
2. Proposition 1 shows that $\mathcal{E}^{\leq s}(M)$ only depends on the ideal I of polynomial functions that vanish on M and that $\mathcal{E}^s(M)$ only depends on the biggest homogeneous ideal contained in I, i.e. the ideal generated by the homogeneous polynomial functions on Mat_n that vanish on M.
3. Assume that $k = K$ is algebraically closed. Let G be a closed subgroup of GL_n and let A be the algebra of polynomial functions on G. Then the representations of the algebra $\mathcal{E}^{\leq s}(G) = (A^{\leq s})^*$ are precisely the rational representations of G whose coefficients are polynomial functions on G that are of filtration degree $\leq s$. If G contains the nonzero multiples of the identity, then the representations of the algebra $\mathcal{E}^s(G) = (A^s)^*$ are precisely the rational representations of G whose coefficients are homogeneous polynomial functions on

G that are of degree s. In case $G = \mathrm{GL}_n$, $(A^s)^*$ is the well-known *Schur algebra* denoted by $S_K(n, s)$ in [11]. Similar remarks apply to a closed submonoid M of Mat_n. Then the coefficients of a rational representation of M are automatically polynomial, since all regular functions on M are polynomial.

We now come to the problem of finding defining equations for the enveloping algebras $\mathcal{E}^s(M)$ and $\mathcal{E}^{\leq s}(M)$. We are only interested in defining equations within $\mathcal{E}^s(\mathrm{Mat}_n))$ and $\mathcal{E}^{\leq s}(\mathrm{Mat}_n)$. By Proposition 1(iii) and (iv) one can then obtain a complete set of defining equations (within $\mathrm{End}_k(\otimes^s V)$ and $\oplus_{r=0}^s \mathrm{End}_k(\otimes^r V)$) by adding the equations of "bisymmetry". See [3, (30)] or [18, III (6.8)]. Statement (i) of the next corollary is a formalisation of an idea of Weyl; see [18, p 142].

Corollary. *Let M be a submonoid of Mat_n, let I be the ideal of polynomial functions on Mat_n that vanish on M and let I_{hom} be the biggest homogeneous ideal contained in I. Furthermore, let $g_1, \dots, g_k$ be nonzero elements of I and let $h_1, \dots, h_l$ be nonzero homogeneous elements of I_{hom}. Denote the isomorphism $k[\mathrm{Mat}_n]^{\leq s} \to \mathcal{E}^{\leq s}(\mathrm{Mat}_n)^*$ by η and the isomorphism $k[\mathrm{Mat}_n]^s \to \mathcal{E}^s(\mathrm{Mat}_n)^*$ by θ. Then*

(i) The elements $g_1, \dots, g_k$ are generators of I with property (1) if and only if for each $s \geq 1$ the functionals $\eta(g_i m_i)$, where the m_i are arbitrary monomials in the matrix entries of degree $\leq s - \deg(g_i)$, define the algebra $\mathcal{E}^{\leq s}(M)$.

(ii) The elements $h_1, \dots, h_k$ are generators of I_{hom} if and only if for each $s \geq 1$ the functionals $\theta(h_i m_i)$, where the m_i are arbitrary monomials in the matrix entries of degree $s - \deg(h_i)$, define the algebra $\mathcal{E}^s(M)$.

Proof. Note that I is a proper ideal, since M is nonempty. Denote the subspace of I that consists of the elements of I of degree $\leq s$ by $I^{\leq s}$. Then the g_i are generators of I with property (1) if and only if for each $s \geq 1$ the polynomials $g_i m_i$, where the m_i are arbitrary monomials in the matrix entries of degree $\leq s - \deg(g_i)$, span $I^{\leq s}$. Assertion (i) now follows from the fact that, because of the commutativity of (2), η maps $I^{\leq s}$ to the space of linear functionals on $\mathcal{E}^{\leq s}(\mathrm{Mat}_n)$ that vanish on $\mathcal{E}^{\leq s}(M)$. Assertion (ii) is proved in a similar way. □

Remark. In [8, Thm.'s 9.5a) and 10.5a)] Doty determines generators for the defining ideals of the symplectic and orthogonal monoids in characteristic zero by a method which is essentially assertion (ii) of the above corollary. In characteristic zero one can namely use the defining equations of the symplectic and orthogonal enveloping algebras in $\mathrm{End}_k(\otimes^s V)$ as obtained by Brauer in [3, (46),(47)].

In the remainder of this note we assume that $n = 2m$ is an even integer. We first intoduce some notation for the symplectic group which closely follows that of [14]. Let $i \mapsto i'$ be the involution of $\{1, \dots, n\}$ defined by $i' := n + 1 - i$. Set $\epsilon_i = 1$ if $i \leq m$ and $\epsilon_i = -1$ if $i > m$ and define the $n \times n$-matrix J with coefficients in k by $J_{ij} = \delta_{ij'}\epsilon_i$. So

$$J = \begin{bmatrix} & & & & & 1 \\ & 0 & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & 0 & \\ -1 & & & & & \end{bmatrix}.$$

On V we define the nondegenerate symplectic form $\langle\ ,\ \rangle$ by

$$\langle u, v\rangle := u^T J v = \sum_{i=1}^{n} \epsilon_i u_i v_{i'}\ .$$

The *symplectic group* $\mathrm{Sp}_n = \mathrm{Sp}_n(k)$ is defined as the set of $n \times n$-matrices over k that satisfy $A^T J A = J$, i.e. the matrices for which the corresponding endomorphism of V preserves the form $\langle\ ,\ \rangle$. Clearly those matrices are invertible and Sp_n is a subgroup of GL_n. Furthermore $\mathrm{Sp}_n(K)$ is an algebraic group which is connected, semisimple and defined and split over the prime field. This implies that all root subgroups $U_\alpha(K)$ with respect to some k-split maximal torus are defined over k. Since clearly $U_\alpha(k)$ is dense in $U_\alpha(K)$, we must have that Sp_n is dense in $\mathrm{Sp}_n(K)$.

Note that $A^T J A$ consists of the scalar products of the columns of A with each other and that AJA^T consists of the scalar products of the rows of A with each other. An easy calculation shows that the condition $A^T J A = J$ is equivalent to the condition $AJA^T = J$. We denote the $(i, j)^{\mathrm{th}}$ entry function on Mat_n by x_{ij}. Define

$$g_{ij} := \sum_{l=1}^{n} \epsilon_l x_{li} x_{l'j} \quad \text{and} \quad \overline{g}_{ij} := \sum_{l=1}^{n} \epsilon_l x_{il} x_{jl'}. \tag{3}$$

The condition $A^T J A = J$ means that we require the functions $g_{ij} - \delta_{ij'}\epsilon_i$, $i < j$, to vanish on A. The condition $AJA^T = J$ means that we require the functions $\overline{g}_{ij} - \delta_{ij'}\epsilon_i$, $i < j$, to vanish on A. We call the ideal of polynomial functions on Mat_n that vanish on Sp_n the *symplectic ideal*. It is well known that both sets of functions separately generate the symplectic ideal in case k is algebraically closed. In view of the density of Sp_n in $\mathrm{Sp}_n(K)$ this must then also hold for an arbitrary infinite field k.

In the sequel we will also need the *symplectic monoid* SpM_n and the *symplectic similitude group* GSp_n as introduced in [8]; see also [14] and [9]. The symplectic monoid SpM_n is defined as the set of matrices A for which there exists a scalar $d(A) \in k$ such that $A^T J A = AJA^T = d(A)J$. Note that if $d(A) \neq 0$, $A^T J A = d(A)J$ is equivalent to $AJA^T = d(A)J$. Clearly SpM_n is a submonoid of Mat_n, in fact it is the set of k-defined points of $\mathrm{SpM}_n(K)$ which is a k-defined closed submonoid of $\mathrm{Mat}_n(K)$. This follows from [14, Cor. 6.2]. The function d is a polynomial function on SpM_n, it is called the *coefficient of dilation*. We have for all $i \in \{1, \ldots, n\}$

$$d = \epsilon_i g_{ii'} = \epsilon_i \overline{g}_{ii'} \ \text{ on } \mathrm{SpM}_n.$$

The *symplectic similitude group* GSp_n is defined as the group of invertible elements in SpM_n. We have $\mathrm{GSp}_n(K) = K^\times \mathrm{Sp}_n(K)$ and GSp_n is Zariski dense in $\mathrm{GSp}_n(K)$ and $\mathrm{SpM}_n(K)$. So a polynomial function on Mat_n vanishes on GSp_n if and only if it vanishes on SpM_n and the ideal of $k[\mathrm{Mat}_n]$ that consists of these functions is the biggest homogeneous ideal contained in the symplectic ideal.

We will now give defining equations for $\mathcal{E}^{\le s}(\mathrm{Sp}_n)$. As stated before the Corollary to Proposition 1, we are only interested in defining equations within the algebra $\mathcal{E}^{\le s}(\mathrm{Mat}_n)$. This algebra consists of $(s+1)$-tuples $(A_0, \dots, A_s)$ of endomorphisms with $A_r \in \mathrm{End}_k(\otimes^r V)$ bisymmetric for all $r \in \{0, \dots, s\}$. The standard basis $(e_1, \dots, e_n)$ of $V = k^n$ gives bases for the vector spaces $\otimes^r V$. The entry of index $((i_1, \dots, i_r), (j_1, \dots, j_r))$ of the matrix of A_r with respect to these bases is denoted by $a_{i_1 \cdots i_r, j_1 \cdots j_r}$.

The first statement in (i) of the theorem below was already remarked by S. Oehms [14, p 38].

Theorem 1. *The following holds*

(i) $\mathrm{gr}(I)$ *is generated by the elements* g_{ij} *and* $\overline{g}_{ij}$*,* $i < j$*. Furthermore,* $\mathrm{gr}(k[\mathrm{Sp}_n])$ *is isomorphic to the algebra of polynomial functions on the set of* $n \times n$*-matrices over* k *whose row and column space are totally singular and this algebra is an integral domain.*

(ii) The sets of elements $g_{ij} - \delta_{ij'}\epsilon_i$ *and* $\overline{g}_{ij} - \delta_{ij'}\epsilon_i$*,* $i < j$*, form together a set of generators of the symplectic ideal with property* (1)*.*

(iii) The symplectic enveloping algebra in $\mathrm{End}_k(T^{\le s}(V))$ *is defined by the following equations for* $r = 2, \dots, s$

$$\sum_{l=1}^{n} \epsilon_l a_{ll' i_3 \cdots i_r, j_1 \cdots j_r} = \delta_{j_1, j_2'} \epsilon_{j_1} a_{i_3 \cdots i_r, j_3 \cdots j_r}\,, \tag{4}$$

$$\sum_{l=1}^{n} \epsilon_l a_{i_1 \cdots i_r, ll' j_3 \cdots j_r} = \delta_{i_1, i_2'} \epsilon_{i_1} a_{i_3 \cdots i_r, j_3 \cdots j_r}\,. \tag{5}$$

Proof. (i). Let J be the ideal of $k[\mathrm{Mat}_n]$ generated by the elements g_{ij} and $\overline{g}_{ij}$, $i < j$. Then the zero set of J is the set of $n \times n$-matrices over k whose row and column space are totally singular. It is also the zero set of d in SpM_n. In fact it follows from [14, Cor. 6.2] that $k[\mathrm{SpM}_n]/(d) \cong k[\mathrm{Mat}_n]/J$. This result also implies that $k[\mathrm{SpM}_n]/(d-1) \cong k[\mathrm{Mat}_n]/I = k[\mathrm{Sp}_n]$. Since d is nonzero and homogeneous of degree $2 > 0$, we have $\mathrm{gr}(d-1) = d$ and $\mathrm{gr}((d-1)) = (d)$. Therefore

$$k[\mathrm{Mat}_n]/J \cong k[\mathrm{SpM}_n]/(d) \cong \mathrm{gr}(k[\mathrm{Sp}_n]) \cong k[\mathrm{Mat}_n]/\mathrm{gr}(I).$$

Since the composite is the epimorphism given by the inclusion $J \subseteq \mathrm{gr}(I)$, we must have $J = \mathrm{gr}(I)$.

Let $\mathcal{V}$ be the variety of $n \times n$-matrices over K whose row and column space are totally singular. Then we have an epimorphism $K[\mathrm{SpM}_n]/(d) \twoheadrightarrow K[\mathcal{V}]$ of graded K-algebras. To show that it is an isomorphism it suffices to show that for each l the dimensions of the graded pieces of degree l of both algebras are equal. By [14, Thm. 6.1] the l^{th} graded piece of $K[\mathrm{SpM}_n]/(d)$ has a basis consisting of all bideterminants $(T_1|\,T_2)$, where T_1 and T_2 are symplectic standard tableaux

in the sense of King of the same shape λ which is a partition of l with at most m parts (notation: $\lambda \vdash_m l$). So its dimension is $\sum_{\lambda \vdash_m l} N_\lambda^2$, where N_λ is the number of symplectic standard tableaux in the sense of King of shape λ. By [5, Thm. 6.1] the l^{th} graded piece of $K[\mathcal{V}]$ has a basis consisting of all bideterminants $(T_1|\,T_2)$, where T_1 and T_2 are symplectic standard tableaux in the sense of De Concini of the same shape $\lambda \vdash_m l$.[3] So its dimension is $\sum_{\lambda \vdash_m l} {N'_\lambda}^2$, where N'_λ is the number of symplectic standard tableaux in the sense of De Concini of shape λ. Let $Y(\lambda)$ be the irreducible $\mathrm{Sp}_n(\mathbb{C})$-module associated to $\lambda \vdash_m l$. Then $N_\lambda = \dim(Y(\lambda)) = N'_\lambda$, by [1] or [7, Thm. 2.3b] and [5, Thm.'s 4.8, 4.10, Prop. 4.10]. Note that we now also know that $\mathcal{V}$ is defined over k.

We will now show that $\mathcal{V}$ is irreducible. Let $\mathcal{V}_0$ be the set of $n \times n$-matrices whose row and column space lie in the K-span V_0 of $e_1, \dots, e_m$. Note that V_0 is a maximal totally singular subspace of $V(K) = K^n$ and that $\mathcal{V}_0$ is a k-defined vector subspace of $\mathrm{Mat}_n(K)$, where $\mathrm{Mat}_n(K)$ has the standard k-structure. Now consider the morphism $\mathrm{Sp}_n(K) \times \mathcal{V}_0 \times \mathrm{Sp}_n(K) \to \mathrm{Mat}_n(K)$ given by $(A, S, B) \mapsto ASB$. Clearly its image lies in $\mathcal{V}$ and by Witt's Lemma its image is equal to $\mathcal{V}$. Since $\mathrm{Sp}_n(K)$ and $\mathcal{V}_0$ are irreducible we must have that $\mathcal{V}$ is irreducible. Furthermore the set of k-defined points is dense in $\mathcal{V}$, since this holds for $\mathrm{Sp}_n(K)$ and $\mathcal{V}_0$ and our morphism is defined over k. Note that, since $K[\mathcal{V}]$ is an integral domain, d generates a prime ideal in $K[\mathrm{SpM}_n]$ and therefore also in $k[\mathrm{SpM}_n]$.

(ii). This follows from (i) and Lemma 1.

(iii). This follows immediately from (ii) and (i) of the Corollary to Proposition 1. □

Remarks 2. 1. The equations in (iii) are the same as those obtained by Weyl [18, p 174] in characteristic zero.

2. The equations for $\mathcal{E}^s(\mathrm{Sp}_n)$ that Brauer [3, (47)] found in characteristic zero also define this algebra in positive characteristic. This can be shown as follows. Let I be the symplectic ideal. Then [14, Cor. 6.2] gives homogeneous generators of I_{hom}.[4] Here one can avoid the FRT-construction by simply taking [14, (13)] as the definition of $A^s_R(n)$. The stability under base change is then trivial and the proofs of [14, Thm 6.1 and Cor. 6.2] still apply. Now (ii) of the Corollary to Proposition 1 gives the following equations for $\mathcal{E}^s(\mathrm{Sp}_n)$ within $\mathcal{E}^s(\mathrm{Mat}_n)$:

$$\delta_{i_1,i'_2}\epsilon_{i_1}\sum_{l=1}^{n}\epsilon_l a_{ll' i_3\cdots i_r, j_1\cdots j_r} = \delta_{j_1,j'_2}\epsilon_{j_1}\sum_{l=1}^{n}\epsilon_l a_{i_1\cdots i_r, ll' j_3\cdots j_r}.$$

2. Infinitesimal invariants in $K[G]$

In this section we assume that $k = K$ is algebraically closed. Furthermore, G is a connected reductive algebraic group over K and $\mathfrak{g} = \mathrm{Lie}(G)$ is its Lie algebra. The conjugation action of G on itself induces an action of G on $K[G]$, the algebra of regular functions on G. We will refer to this action and its derived

[3]De Concini uses different terminology, but it is clear that his condition of symplectic standardness is imposed on T_1 and T_2 separately.

[4]In [14, (14)], $f_{ll'} - f_{kk'}$ should be replaced by $\epsilon_k f_{ll'} - \epsilon_l f_{kk'}$ (or by $\epsilon_l f_{ll'} - \epsilon_k f_{kk'}$).

$\mathfrak{g}$-action as *conjugation* actions. The conjugation action of G on $K[G]$ is by algebra automorphisms, so the conjugation action of $\mathfrak{g}$ on $K[G]$ is by derivations. This implies that the space $K[G]^{\mathfrak{g}} = \{f \in K[G] \,|\, x \cdot f = 0 \text{ for all } x \in \mathfrak{g}\}$ of *infinitesimal invariants* is a subalgebra of $K[G]$. Note that $K[G]^G \subseteq K[G]^{\mathfrak{g}}$ and that $K[G]^p \subseteq K[G]^{\mathfrak{g}}$ if K is of characteristic $p > 0$.

Theorem 2. *Assume that K is of characteristic $p > 0$ and that the derived group of G is simply connected. Then the invariant algebra $K[G]^{\mathfrak{g}}$ is a unique factorisation domain. Its irreducible elements are the irreducible elements of $K[G]$ that are invariant under $\mathfrak{g}$ and the p-th powers of the irreducible elements of $K[G]$ that are not invariant under $\mathfrak{g}$.*

Proof. By Remark 3 in [16] we have to prove Proposition 1 in that paper for Sp_n. In fact we may assume that $n \geq 2$ and that K is of characteristic 2, but we will not use this. The statement of Proposition 1 in [16] is:

Let $f \in K[G]$ be a regular function. If the ideal $K[G]f$ is stable under the conjugation action of $\mathfrak{g}$ on $K[G]$, then f is $\mathfrak{g}$-invariant.

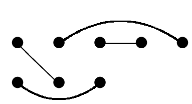
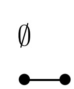
By Proposition 2 and Lemma 2 in [16] it is sufficient to show that, for the filtration that $K[\mathrm{Sp}_n]$ inherits from $K[\mathrm{Mat}_n]$, the associated graded algebra is an integral domain. The point is that one can then use a (filtration) degree argument. The result now follows immediately from Theorem 1(ii). □

Proposition 2. *The algebras $K[\mathrm{Sp}_n]$, $K[\mathrm{GSp}_n]$ and $K[\mathrm{SpM}_n]$ are unique factorisation domains.*

Proof. Since the derived group $D\mathrm{GSp}_n$ of GSp_n equals Sp_n which is simply connected, we have by [16, Cor. to Thm. 1] that $K[\mathrm{Sp}_n]$ and $K[\mathrm{GSp}_n]$ are UFD's. We have $K[\mathrm{SpM}_n][d^{-1}] = K[\mathrm{GSp}_n]$ which is a UFD. Furthermore d generates a prime ideal in $K[\mathrm{SpM}_n]$ by the proof of Theorem 1(i). So $K[\mathrm{SpM}_n]$ is a UFD by Nagata's Lemma; see e.g. [10, Lemma 19.20]. □

3. A double centraliser theorem for the symplectic group

We begin by defining a version of the symplectic Brauer algebra. For each integer $r \leq s$ we fix r vector symbols $x_1, \ldots, x_r$ and r covector symbols $y_1, \ldots, y_r$.[5] To ease notation we did not give these symbols the extra index r that indicates to which integer $r \leq s$ they are associated. In fact one may, as the notation suggests, take the r-symbols as the first r of the s-symbols. Let u and v be nonnegative integers $\leq s$ such that $u \equiv v \pmod 2$, that is, such that $u + v$ is even. A (u, v)-*diagram* is a matching of the $u + v$ symbols $y_1, \ldots, y_u, x_1, \ldots, x_v$ in pairs. Such a diagram is depicted as a graph whose vertices are arranged in two rows. The top row consists of u vertices representing, from left to right, the u covector symbols $y_1, \ldots, y_u$ and the bottom row consists of v vertices representing the v vector symbols $x_1, \ldots, x_v$. An empty row is represented by $\emptyset$. Two vertices are joined if the corresponding symbols are matched. So for example

 and  (6)

[5]elements of V are called *vectors* and elements of V^* are called *covectors*.

are a $(5,3)$ and a $(0,2)$-diagram.

If an edge e in a (u,v)-diagram is horizontal, then its left endpoint is called its *initial point* and its right endpoint is called its *terminal point*. If e is not horizontal, then its endpoint in the top row is called its initial point and its endpoint in the bottom row is called its terminal point.

Let $t\in k$. We now define the k-algebra $\mathfrak{B}_{\le s}(t)$. It has all (u,v)-diagrams, $0\le u,v\le s$, $u\equiv v \pmod 2$, as a k-basis and its dimension is $\sum_{u=1}^{s}\sum_{v=1}^{s} N_{uv}$, where

$$N_{uv}=\begin{cases}(u+v-1)(u+v-3)\cdots 3{\cdot}1=\frac{(u+v)!}{2^{(u+v)/2}((u+v)/2)!} & \text{if } u\equiv v \pmod 2,\\ 0 & \text{otherwise.}\end{cases}$$

The multiplication of diagrams is defined as follows. Let D_{uv} be a (u,v)-diagram and let $D_{u'v'}$ be a (u',v')-diagram. Then we define

$$D_{uv}D_{u'v'}=\begin{cases}\mathrm{sgn}(D_{uv},D_{u'v'})t^{\gamma(U)}D_{uv'} & \text{if } v=u',\\ 0 & \text{if } v\ne u'.\end{cases}$$

Here, as in [3] and [12], U is the graph which is obtained by putting D_{uv} on top of $D_{u'v'}$ and identifying the vertices of the bottom row of D_{uv} with those of the top row of $D_{u'v'}$, $\gamma(U)$ is the number of cycles in U and $D_{uv'}$ is the (u,v')-diagram obtained from U by taking the vertices from top and bottom row in U and matching two vertices if there is a path between them in U.

Our definition of the sign $\mathrm{sgn}(D_{uv},D_{u'v'})$ of D_{uv} over $D_{u'v'}$ does not agree with [12, p 411]. The graph U consists of $(u+v')/2$ paths and $\gamma(U)$ cycles that are of even length and consist entirely of edges in the middle row of U. A path in U either consists of one horizontal edge in the top or bottom row of U or it has precisely two non-horizontal edges and all its horizontal edges are in the middle row of U. A path of even length has one endpoint in the top row of U and the other endpoint in the bottom row of U. A path of odd length has its endpoints both in the top row of U or both in the bottom row of U.

Every path P has a "natural" orientation. If P has even length, then the endpoint in the top row is the initial point and the endpoint in the bottom row is the terminal point. If P has odd length, then the leftmost endpoint of P is the initial point and the other endpoint is the terminal point. For a path P or an oriented cycle P we denote the number of edges of P in the middle row of U that are traversed from left to right by p_{lr} and the number of edges of P in the middle row of U that are traversed from right to left by p_{rl}. For a cycle or a path P we define

$$\mathrm{sgn}(P)=\begin{cases}(-1)^{|p_{lr}-p_{rl}|/2} & \text{if } p_{lr}+p_{rl} \text{ is even,}\\ (-1)^{|p_{lr}-p_{rl}-1|/2} & \text{if } p_{lr}+p_{rl} \text{ is odd.}\end{cases}$$

Note that for a cycle this does not depend on the orientation. We now define $\mathrm{sgn}(D_{uv},D_{u'v'})$ as the product of the signs of all paths and cycles in U.

Below we will define a natural representation of $\mathfrak{B}_{\le s}(n)$ which is the motivation for the definition of the multiplication. We will see later that the multiplication of $\mathfrak{B}_{\le s}(t)$ is associative.

With each (u,v)-diagram D we can associate a $(u+v)$-multilinear function $F(D)$ on $\bigoplus^u V^* \oplus \bigoplus^v V$ as follows. First we observe that the form $\langle\ ,\ \rangle$ defines an isomorphism $V \cong V^*$, unique up to sign, and therefore a unique symplectic form on V^* which we also denote by $\langle\ ,\ \rangle$. Furthermore we put $\langle y, x\rangle = y(x) = -\langle x, y\rangle$ for $y \in V^*$ and $x \in V$. If e is an edge in D whose initial point has label z_1 and whose terminal point has label z_2, then we put $\langle e\rangle = \langle z_1, z_2\rangle$. Now we define

$$F(D) = \prod_{e\in D} \langle e\rangle. \tag{7}$$

For example, for the first diagram D in (6) we have

$$F(D) = \langle y_1, x_2\rangle\langle y_2, y_5\rangle\langle y_3, y_4\rangle\langle x_1, x_3\rangle.$$

Using the k-vector space isomorphisms

$$\mathrm{End}_k(T^{\le s}(V)) \cong \bigoplus_{0\le u,v\le s} \mathrm{Hom}_k(\otimes^v V, \otimes^u V) \tag{8}$$

and

$$\mathrm{Hom}_k(\otimes^v V, \otimes^u V) \cong (\otimes^u V) \otimes (\otimes^v V)^* \cong \big((\otimes^u V^*) \otimes (\otimes^v V)\big)^*, \tag{9}$$

we can now associate to each (u,v)-diagram D an endomorphism $E(D)$ of $T^{\le s}(V)$ as follows. We take $E(D)$ to be the endomorphism of $T^{\le s}(V)$ corresponding to $F(D) \in \big((\otimes^u V^*) \otimes (\otimes^v V)\big)^*$. Let $(e_1,\dots,e_n)$ be the standard basis of $V = k^n$. Then we have a dual basis $(e_1^*,\dots,e_n^*)$ and also bases of the vector spaces $\otimes^r V$. The matrix $M(D)$ of $E(D) \in \mathrm{Hom}_k(\otimes^v V, \otimes^u V)$ with respect to these bases is then given by

$$M(D)_{(i_1,\dots,i_u),(j_1,\dots,j_v)} = F(D)(e_{i_1}^*,\dots,e_{i_u}^*, e_{j_1},\dots,e_{j_v}). \tag{10}$$

The linear map $E : \mathfrak{B}_{\le s}(n) \to \mathrm{End}_k(T^{\le s}(V))$ given by the assignment $D \mapsto E(D)$ is a homomorphism of algebras. We indicate a proof.

The isomorphism $\varphi : V^* \to V$ with $\langle \varphi(y), x\rangle = y(x)$ for all $x \in V$ and $y \in V^*$ maps e_i^* to $\epsilon_{i'} e_{i'}$. So we have

$$\langle e_i^*, e_j^*\rangle = \langle e_i, e_j\rangle = \epsilon_i \delta_{ij'}.$$

Now one easily checks that

$$\sum_{l_2,\dots,l_p=1}^{n} \langle e_{l_1}, e_{l_2}\rangle\langle e_{l_2}, e_{l_3}\rangle \cdots \langle e_{l_p}, e_{l_{p+1}}\rangle = \begin{cases} (-1)^{p/2}\langle e_{l_1}^*, e_{l_{p+1}}\rangle & \text{if } p \text{ is even,} \\ (-1)^{(p-1)/2}\langle e_{l_1}, e_{l_{p+1}}\rangle & \text{if } p \text{ is odd.} \end{cases}$$

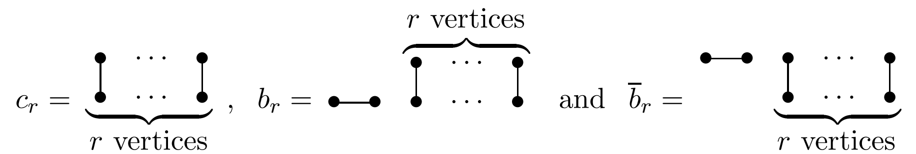
and that for p even

$$\sum_{l_1,\dots,l_p=1}^{n} \langle e_{l_1}, e_{l_2}\rangle\langle e_{l_2}, e_{l_3}\rangle \cdots \langle e_{l_{p-1}}, e_{l_p}\rangle\langle e_{l_p}, e_{l_1}\rangle = (-1)^{p/2} \cdot n\,.$$

If we evaluate an entry of the product $M(D_{uv})M(D_{u'v'})$, $v = u'$, then this entry will be a sum over v indices corresponding to the vertices in the middle row of U. Here we use (7) and (10). Now we can distribute these v sums over the paths and cycles in U. Each path or cycle gets the sums corresponding to the vertices

of the middle row of U that it contains. Then we obtain a product of sums of the above two types where we have to swap the arguments of certain scalar products $\langle e_{l_{i+1}}, e_{l_{i+2}}\rangle$ according to position of the corresponding vertices in the middle row of U. Here an extra sign $(-1)^{p_{lr}}$ comes in. Now it follows that the entry of $M(D_{uv})M(D_{u'v'})$ is equal to the corresponding entry of $M(D_{uv}D_{u'v'})$.

We now define some special elements of $\mathfrak{B}_{\leq s}(t)$.



(11)

Here c_r is defined for $0 \leq r \leq s$ and b_r and $\overline{b}_r$ are defined for $0 \leq r \leq s-2$. The unit element of $\mathfrak{B}_{\leq s}(t)$ is $\sum_{r=0}^{s} c_r$. The subspace $c_r\mathfrak{B}_{\leq s}(t)c_r$ is the span of all (r,r)-diagrams and is closed under multiplication. It has c_r as a unit element and as an algebra it is isomorphic to the symplectic Brauer algebra $\mathfrak{B}_r(t)$ which was introduced for $t = n$ in [3]. We identify $\mathfrak{B}_r(t)$ with this subspace of $\mathfrak{B}_{\leq s}(t)$. Then $\bigoplus_{r=0}^{s}\mathfrak{B}_r(t)$ is identified with a subalgebra with unit of $\mathfrak{B}_{\leq s}(t)$. We obtain a natural embedding $k\langle \mathrm{Sym}_r\rangle \subseteq \mathfrak{B}_r(t)$ by assigning to each $\pi \in \mathrm{Sym}_r$ the (r,r)-diagram in which x_i is matched with y_{π_i}. The action of Sym_r on $\otimes^r V$ that it inherits from $\mathfrak{B}_r(n)$ is the same as the action mentioned in Section 1.

The algebra $\mathfrak{B}_{\leq s}(t)$ is generated by the elements b_r, $\overline{b}_r$, $0 \leq r \leq s-2$, c_0, c_1 and the elements of each $\mathfrak{B}_r(t)$, $2 \leq r \leq s$, that correspond to the elementary transpositions in Sym_r.

In the theorem below we will consider the representation of $\mathfrak{B}_{\leq s}(n)$ on $T^{\leq s}(V)$ as given by the homomorphism E defined above. First we make a preliminary observation. Let $B \in \mathrm{Hom}_k(V\otimes V, k) \subseteq \mathrm{End}_k(T^{\leq s}(V)$ be the endomorphism that maps $x_1 \otimes x_2$ to $\langle x_1, x_2\rangle$ and let $\overline{B} \in \mathrm{Hom}_k(k, V\otimes V) \subseteq \mathrm{End}_k(T^{\leq s}(V)$ be the endomorphism that maps 1 to $\sum_{i=1}^{n}\epsilon_i e_i \otimes e_{i'}$. Then we have

$$E(b_r) = B \otimes \mathrm{id}_{\otimes^r V} \quad\text{and}\quad E(\overline{b}_r) = \overline{B}\otimes \mathrm{id}_{\otimes^r V}. \tag{12}$$

Note furthermore that $E(c_r)$ is just the projection of $T^{\leq s}(V)$ onto $\otimes^r V$.

Theorem 3. *The following holds.*

(i) $\mathrm{End}_{\mathrm{Sp}_n}(T^{\leq s}(V))$ *coincides with the image of* $\mathfrak{B}_{\leq s}(n)$ *in* $\mathrm{End}_k(T^{\leq s}(V))$.
(ii) $\mathrm{End}_{\mathfrak{B}_{\leq s}(n)}(T^{\leq s}(V))$ *is the enveloping algebra of* Sp_n *in* $\mathrm{End}_k(T^{\leq s}(V))$.
(iii) If $m \geq s$, *then the homomorphism* $\mathfrak{B}_{\leq s}(n) \to \mathrm{End}_k(T^{\leq s}(V))$ *is injective.*
(iv) If $m < s$, *then the homomorphism* $\mathfrak{B}_s(n) \to \mathrm{End}_k(\otimes^s V)$ *is not injective.*

Proof. (i). We follow Brauer's method as adjusted to our situation by Weyl, see [18, V.2, p 141-142]. There is a natural action of Sp_n on $\mathrm{End}_k(T^{\leq s}(V))$ and it is clear that $\mathrm{End}_{\mathrm{Sp}_n}(T^{\leq s}(V))$ consists of the Sp_n-invariant elements of $\mathrm{End}_k(T^{\leq s}(V))$. The vector spaces in (8) and (9) all have a natural Sp_n-action and the isomorphisms there are Sp_n-equivariant. It therefore suffices to show that for $u, v \in \{1,\dots,s\}$, $\big((\otimes^u V^*)\otimes(\otimes^v V)\big)^{*\,\mathrm{Sp}_n}$ is spanned by the multilinear functions $F(D)$, where D is a (u,v)-diagram and F is given by (7).

Let $u, v \in \{1,\dots,s\}$ and put $w = u+v$. We will identify V^* with V by means of the isomorphism $\varphi : V^* \xrightarrow{\sim} V$. This means that the y_i are now vector

variables. We put $z_i = y_i$ for $i \in \{1,\dots,u\}$ and $z_i = x_{i-u}$ for $i \in \{u+1,\dots,w\}$. Since k is infinite, $k[\oplus^w V]$ can be identified with the polynomial ring in the components of the z_i. If $f \in k[\oplus^w V]$ is Sp_n-invariant, then it is also $\mathrm{Sp}_n(K)$-invariant as an element of $K\otimes_k k[\oplus^w V]$, since Sp_n is dense in $\mathrm{Sp}_n(K)$. But then f is a formal invariant in the definition of [6, §2], see e.g. [13, Remark I.2.8]. We can now apply the first fundamental theorem of invariant theory for the symplectic group, [6, Thm. 6.6]. This gives us that $k[\oplus^w V]^{\mathrm{Sp}_n}$ is generated as a k-algebra by the scalar products $\langle z_i, z_j\rangle$, $1 \le i < j \le w$. It follows immediately that $(\otimes^w V)^{*\,\mathrm{Sp}_n}$ is spanned by the monomials in the $\langle z_i, z_j\rangle$, $1 \le i < j \le w$, in which each z_i occurs exactly once. These monomials are precisely the $F(D)$, where D is a (u,v)-diagram.
(ii). We have $k\langle\mathfrak{S}\rangle \subseteq \mathfrak{B}_{\le s}(n)$, where $\mathfrak{S}$ is as in Proposition 1(iv). So, as in the proof of that result, we have that $\mathrm{End}_{\mathfrak{B}_{\le s}(n)}(T^{\le s}(V))$ consists of $(s+1)$-tuples of bisymmetric substitutions. Using (12) one easily checks that the condition of commuting with $E(b_r)$ is given by the equation (4) and that (5) gives the condition of commuting with $E(\overline{b}_r)$. By Theorem 1(iii) these are precisely the equations that define $\mathcal{E}^{\le s}(\mathrm{Sp}_n)$.

The arguments in the proofs of (iii) and (iv) below are very similar to those for the orthogonal group in [18, p 149]. Recall that the *Pfaffian* of an alternating $2r \times 2r$-matrix A is defined by

$$\mathrm{Pf}(A) = \sum_{\pi} \mathrm{sgn}(\pi) \prod_{i=1}^{r} a_{\pi_{2i-1},\pi_{2i}}\ , \tag{13}$$

where the sum is over all permutations π of $\{1,2,\dots,2r\}$ with $\pi_1 < \pi_3 < \cdots < \pi_{2r-1}$ and $\pi_{2i-1} < \pi_{2i}$ for all $i \in \{1,\dots,r\}$.
(iii). By the isomorphisms (8) and (9) it is sufficient to show that for each $u, v \in 1,\dots,s$ with $u \equiv v \pmod 2$, the multilinear functions $F(D)$, D a (u,v)-diagram, are linearly independent. We use the identification $V^* \cong V$ and the notation of (i). As we have seen in (i) the $F(D)$ are the monomials in the $\langle z_i, z_j\rangle$, $1 \le i < j \le w$ in which each z_i occurs exactly once. The second fundamental theorem for the symplectic group, [6, Thm. 6.7], says that the ideal of relations between the $\langle z_i, z_j\rangle$, $1 \le i < j \le w$ is generated by the Pfaffians of the principal submatrices of size $n+1$ of the $w \times w$ alternating matrix $\langle z_i, z_j\rangle_{1\le i,j\le w}$.[6] Since, by assumption, $w \le 2s \le n$, there are no such submatrices, that is, the $\langle z_i, z_j\rangle$, $1 \le i < j \le w$ are algebraically independent. This means that the monomials in the $\langle z_i, z_j\rangle$, $1 \le i < j \le w$ are linearly independent.
(iv). From (13) it is clear that $\mathrm{Pf}(A)$ is a signed sum of distinct monomials in the a_{ij}, $1 \le i < j \le 2r$, in which each number $i \in \{1,2,\dots,2r\}$ occurs precisely once either as row or as column index. Since $n < 2s$, we must have that the alternating $2s$-multilinear map $\mathrm{Pf}(\langle z_i, z_j\rangle_{1\le i,j\le 2s})$ on $\oplus^{2s}V$ is zero. This gives us a nontrivial dependence relation between the multilinear functions $F(D)$, D an (s,s)-diagram. □

Corollary. *The algebra $\mathfrak{B}_{\le s}(t)$ is associative for any $t \in k$.*

[6] A principal submatrix is obtained by choosing rows and columns from the same index set.

Proof. First we observe that we can define the Brauer algebra over any commutative ring R and then we have $\mathfrak{B}_{\leq s}(R,t_0)\cong R\otimes_{\mathbb{Z}[t]}\mathfrak{B}_{\leq s}(\mathbb{Z}[t],t)$ for all $t_0\in R$, where we now consider t as an indeterminate and the homomorphism $\mathbb{Z}[t]\to R$ is given by $t\mapsto t_0$. So it suffices to show that $\mathfrak{B}_{\leq s}(\mathbb{Z}[t],t)$ is associative. If $D,D',D''\in\mathfrak{B}_{\leq s}(\mathbb{Z}[t],t)$, then the coefficients of $(DD')D''-D(D'D'')$ with respect to the diagram basis are polynomials in t. By Theorem 3(iii) applied with $k=\mathbb{Q}$, these polynomials vanish whenever we specialise t to an even integer $\geq 2s$. It follows that these polynomials are identically zero. □

Remarks 3. 1. One can also derive the double centraliser theorem for Sp_n acting on $\otimes^s V$ as otained in [9, Prop. 1.3 and Thm 1.4] by our method. Simply combine Remark 2.2 and the arguments of the proof of Theorem 3.
2. Drop the assumption that n is even and assume that the characteristic of k is not 2. Orthogonal versions $\mathfrak{A}_{\leq s}(t)$ and $\mathfrak{A}_s(t)$ of $\mathfrak{B}_{\leq s}(t)$ and $\mathfrak{B}_s(t)$ can also be defined. One just has to omit the sign in the definition of the multiplication. The algebra $\mathfrak{A}_s(t)$ coincides with the orthogonal Brauer algebra defined in [12]. Furthermore the algebas $\mathfrak{A}_{\leq s}(n)$ and $\mathfrak{A}_s(n)$ have natural representations in $T^{\leq s}(V)$ and $\otimes^s V$.

Let I be the orthogonal ideal. In characteristic zero homogeneous generators for I_{hom} are given in [8, Thm. 10.5a)] and generators for I with property (1) are given in [18, Thm. 5.2.C]. The Corollary to Proposition 1 and the proof of Theorem 3 show that, to prove the double centraliser theorem over k for the orthogonal group acting on $\otimes^s V$ and $T^{\leq s}(V)$, it is enough to show that these elements are also generators of I_{hom} and generators for I with property (1) over k.
3. Assume that $k=\mathbb{C}$. Then it can be shown by easy invariant theoretic arguments as in the proof of Theorem 3(iii) and (iv) that the natural representation $\mathfrak{A}_s(n)\to\mathrm{End}_k(\otimes^s V)$ is injective if $n\geq 2s$ and not injective if $n<s$; see [18, p 149]. So, contrary to the symplectic case, these arguments do not give a complete criterion for faithfulness. In [4] it is proved that the natural representation of $\mathfrak{A}_s(n)$ is faithful if and only if $n\geq s$.
4. From [15, Thm.'s 2.11, 4.6, 4.3] it follows that $\mathfrak{A}_s(t)$ has the well-known presentation given in e.g. [9, p 3]. From this one easily deduces a presentation for $\mathfrak{B}_s(t)$ and an isomorphism $\mathfrak{B}_s(t)\cong\mathfrak{A}_s(-t)$. This is in accordance with the results in [12] and [17].

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